Journal of Engineering Mathematics 20 (1985) 181–187

Analysis of fluid equations by group methods

W.F. AMES [1] and M.C. NUCCI [2,3]

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

(Received November 5, 1985)

Summary

Using the machinery of Lie-group analysis several equations arising in fluid mechanics are studied. In particular, the Burgers' equation, the KdV equation, the Hopf equation, the two-dimensional KdV equation and the Lin-Tsien equation are analyzed. In all cases the particular group includes arbitrary functions of time which permit the transformation of time-dependent equations into the corresponding time-independent ones. Infinitely many time-dependent solutions are associated with each steady solution. Some solutions are constructed.

1. Introduction

Perhaps the most widely applicable method for determining analytic solutions of partial differential equations utilizes the underlying (Lie) group structure. The mathematical foundations for the determination of the full group for a system of differential equations can be found in Ames [1], Bluman and Cole [2], and the general theory is found in Ovsiannikov [3]. The determination of the full group requires extremely lengthy calculations. Detailed calculations can be found in Ames [1], Ovsiannikov [3], and for the Navier-Stokes equations in Boisvert [4] (see also Boisvert et al. [5]). Algebraic programming packages for determining these groups have been developed by Schwarz using REDUCE [9], by Roseneau and Schwarzmeier using MACSYMA [10] and CINO in Russia (see Ovsiannikov [3], p. 57). These programs, while very versatile, have difficulties in incorporating arbitrary functions where they arise in the Lie algebra. These arbitrary functions play a fundamental role in the sequel.

In Boisvert et al. [5] the full Lie group leaving the Navier Stokes equations invariant,

$$u_t + uu_x + vu_y + wu_z = -p_x + \mu\nabla^2 u, \tag{1.1}$$

$$v_t + uv_x + vv_y + wv_z = -p_y + \mu\nabla^2 v, \tag{1.2}$$

$$w_t + uw_x + vw_y + ww_z = -p_z + \mu\nabla^2 w, \tag{1.3}$$

$$u_x + v_y + w_z = 0, \tag{1.4}$$

[1] Research supported by U.S. Army Grant DAAG-29-84-K-0083.
[2] Permanent address: Dipartimento di Matematica, Universitá di Perugia, 06100 Perugia, Italy.
[3] Research supported by a NATO-CNR fellowship.

is determined. In the spirit of Lie it is desired to find infinitesimal transformations of the form

$$
\begin{aligned}
t' &= t + \epsilon T(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
x' &= x + \epsilon X(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
y' &= y + \epsilon Y(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
z' &= z + \epsilon Z(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
u' &= u + \epsilon U(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
v' &= v + \epsilon V(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
w' &= w + \epsilon W(t, x, y, z, u, v, w, p) + O(\epsilon^2),\\
p' &= p + \epsilon P(t, x, y, z, u, v, w, p) + O(\epsilon^2),
\end{aligned}
\tag{1.5}
$$

which leave (1.1–1.4) invariant. System (1.5) leaves (1.1–1.4) invariant if and only if (u', v', w', p') is a solution of (1.1′–1.4′) whenever (u, v, w, p) is a solution to (1.1–1.4). By (1.1′–1.4′) is meant the same equations in the primed variables. By extensive analysis it is found that the full Lie group leaving (1.1–1.4) invariant is given by (1.5) with

$$T = \alpha + 2\beta t, \tag{1.6}$$

$$X = \beta x - \gamma y - \lambda z + f(t), \tag{1.7}$$

$$Y = \beta y + \gamma x - \sigma z + g(t), \tag{1.8}$$

$$Z = \beta z + \lambda x + \sigma y + h(t), \tag{1.9}$$

$$U = -\beta u - \gamma v - \lambda w + f'(t), \tag{1.10}$$

$$V = -\beta v + \gamma u - \sigma w + g'(t), \tag{1.11}$$

$$W = -\beta w + \lambda u + \sigma v + h'(t), \tag{1.12}$$

$$P = -2\beta p + j(t) - xf''(t) - yg''(t) - zh''(t), \tag{1.13}$$

where α, β, γ, λ and σ are five arbitrary parameters and $f(t)$, $g(t)$, $h(t)$ and $j(t)$ are arbitrary, sufficiently smooth functions of t.

The arbitrary functions in (1.7–1.9) permit equations (1.1–1.4) to be transformed into their time-independent form. Thus any solution of the steady equations generates an infinity of time-dependent solutions. This idea was exploited in Boisvert et al. [5] and Nucci [6].

2. Burgers' equation

For the Burgers' equation

$$u_t + uu_x = \mu u_{xx} \tag{2.1}$$

the full two-parameter group (α, β) with one arbitrary function ($f(t)$) is

$$T = \alpha + 2\beta t, \quad X = \beta x + f(t), \quad U = -\beta u + f'(t). \tag{2.2}$$

with $\alpha = 1$, $\beta = 0$ the subgroup

$$T = 1, \quad X = f(t), \quad U = f'(t)$$

has the generator

$$QI = \frac{\partial I}{\partial t} + f(t)\frac{\partial I}{\partial x} + f'(t)\frac{\partial I}{\partial u} = 0. \tag{2.3}$$

From the Lagrange equations of (2.3) we have

$$\bar{u} = u - f(t), \quad \bar{x} = x - F(T), \tag{2.4}$$

where $F' = f$. When this transformation is applied to (2.1) there results

$$\bar{u}\bar{u}_{\bar{x}} = \mu \bar{u}_{\bar{x}\bar{x}}, \tag{2.5}$$

that is, the steady equation. One integration gives the integrable Riccati equation

$$U' + U^2 = C, \tag{2.6}$$

where $\bar{u} = -2\mu U$. Finally, setting $U = \psi'/\psi$, (2.6) becomes

$$\psi'' - C\psi = 0.$$

For $C = \alpha^2 > 0$ the solution of (2.1) is

$$u(x, t) = -2\mu\alpha\{1 - R\exp[-2\alpha(x - F(t))]\}$$
$$/\{1 + R\exp[-2\alpha(x - F(t)]\} + f(t).$$

For $C = -\alpha^2 < 0$ the solution for (2.1) is

$$u(x, t) = -2\mu\alpha\frac{\cos[\alpha(x - F(t))] - R\sin[\alpha(x - F(t))]}{\sin[\alpha(x - F(t))] + R\cos[\alpha(x - F(t))]} + f(t)$$

where R is another arbitrary constant. If $C = 0$ the solution is

$$u(x, t) = \frac{2\mu}{2\mu R - (x - F(t))} + f(t).$$

3. The Korteweg-De Vries equation

Under the transformation (2.4) the KdV equation $u_t + uu_x + u_{xxx}$ becomes

$$\bar{u}\bar{u}_{\bar{x}} = \bar{u}_{\bar{x}\bar{x}\bar{x}},$$

which is integrable in terms of elliptic functions since one integration gives

$$U_{xx} - U^2 = C,$$

when $\bar{u} = 2U$.

4. The equation $u_t + uu_x = [\phi(u_x)u_x]_x$

Again the action of (2.4) transforms the equation of the title into

$$\bar{u}\bar{u}_{\bar{x}} = [\phi(\bar{u}_{\bar{x}})\bar{u}_{\bar{x}}]_{\bar{x}}.$$

5. Two-dimensional KdV equation

The full group for the two-dimensional KdV equation (Rogers and Shadwick [7])

$$u_{xxxx} = -u_{xt} - \alpha u_{yy} - 6u_x^2 - 6uu_{xx} \tag{5.1}$$

is calculated, using the notation of (1.5) for t, x, y, and u, to be

$$\begin{aligned} &T = f(t), \\ &X = f'(t)x/3 - f''(t)y^2/6\alpha - g'(t)y/2\alpha + h(t), \\ &Y = 2f'(t)y/3 + g(t), \\ &U = -2f'(t)u/3 + f''(t)x/18 - f'''(t)y^2/36\alpha - g''(t)y/12\alpha + h'(t)/6. \end{aligned} \tag{5.2}$$

In (5.2) the functions $f(t)$, $g(t)$ and $h(t)$ are arbitrary so the Lie algebra is infinite-dimensional with the generator

$$Q = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial u}. \tag{5.3}$$

Moreover, the *specific choice* of the subgroup

$$T = 1, \quad X = -g'y/2\alpha + h(t), \quad Y = g(t), \quad U = -g''y/12\alpha + h'/6 \tag{5.4}$$

gives rise to the characteristic variables

$$\begin{aligned} &y - \bar{y} = G(t), \\ &x - \bar{x} = -gy/2\alpha + \Omega(t), \\ &u - \bar{u} = -g'y/12\alpha + g^2/24\alpha + h/6, \end{aligned} \tag{5.5}$$

where $G' = g$ and $\Omega' = g^2/2\alpha + h$. Under (5.5) equation (5.1) becomes

$$\alpha \bar{u}_{\bar{y}\bar{y}} + 6(\bar{u}_{\bar{x}})^2 + 6\bar{u}\bar{u}_{\bar{x}\bar{x}} + \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \tag{5.6}$$

that is, the time-independent equation. Each solution of (5.6) gives rise to a family of solutions involving three arbitrary functions of time, f, g and h, when $\bar{u}$, $\bar{x}$, and $\bar{y}$ are replaced by their relations from (5.5). To illustrate this idea the full group is generated for equation (5.6) and some exact solutions are obtained in the next section.

6. Solutions for equation (5.6)

The full group for equation (5.6) (we have dropped the bars), in the notation of (1.5) for x, y, and u, is

$$X = c_1 x + c_2, \quad Y = 2c_1 y + c_3, \quad U = -2c_1 u, \tag{6.1}$$

with the three parameters c_1, c_2, c_3. The equations for the invariant surface are obtained from (for $c_1 \neq 0$)

$$\begin{aligned} u &= F(\eta)/2c_1(2c_1 y + c_3), \\ \eta &= (c_1 x + c_2)/(2c_1 y + c_3)^{1/2}, \end{aligned} \tag{6.2}$$

where F satisfies the ordinary differential equation

$$c_1^3 F^{(iv)} + 3FF'' + \alpha c_1 \eta^2 F'' + 3(F')^2 + 7c_1 \alpha \eta F' + 8\alpha c_1 F = 0. \tag{6.3}$$

Two solutions of equation (6.3) are

$$F(\eta) = -4\alpha c_1 \eta^2/3 \tag{6.4}$$

and

$$F(\eta) = -4c_1^3 \eta^{-2}. \tag{6.5}$$

The solution of equation (5.1) resulting from (6.4) is

$$u = -\frac{2\alpha\left\{c_1\left[x + \frac{g}{2\alpha}y - \Omega\right] + c_2\right\}^2}{3\{2c_1[y - G(t)] + c_3\}^2} - \frac{g'}{12\alpha}y + \frac{g^2}{24\alpha} + \frac{h}{6},$$

and that resulting from (6.5) is

$$u = -\frac{2c_1^2}{\left\{c_1\left[x + \frac{g}{2\alpha}y - \Omega(t)\right] + c_2\right\}^2} - \frac{g'}{12\alpha}y + \frac{g^2}{24\alpha} + \frac{h}{6},$$

where $\Omega' = g^2/24\alpha + h$ and $G' = g$.

7. An interesting subgroup of (5.2)

With the choice $f(t)=t^3$, $g(t)=h(t)=0$ in (5.2) the group becomes

$$T=t^3,\quad X=t^2x-ty^2/2,\quad Y=2t^2y,\quad U=-2t^2u+tx/3-y^2/6\alpha. \tag{7.1}$$

From (7.1) the equations for the invariant surface are found from

$$\eta=y/t^2,\quad \xi=x/t+y^2/2\alpha t^2 \tag{7.2}$$

and

$$u=\xi/6-\eta^2t^2/12\alpha+F(\eta,\,\xi)/t^2, \tag{7.3}$$

where $F(\eta,\,\xi)$ satisfies the equation

$$\alpha F_{\eta\eta}+(3F^2)_{\xi\xi}+F_{\xi\xi\xi\xi}=0. \tag{7.4}$$

But, of course, this is equation (5.6) with $\eta=\bar{y}$, $\xi=\bar{x}$ and $F=\bar{u}$. Thus two solutions are available, i.e.,

$$u(x,\,t)=\frac{x}{6t}-\frac{2\alpha\left[c_1xt+c_1y^2/2\alpha+t^2c_2\right]^2}{3t^2\left[2c_1y+t^2c_3\right]^2}$$

and

$$u(x,\,t)=\frac{x}{6t}-\frac{2c_1^2t^2}{\left(c_1xt+c_1y^2/2\alpha+t^2c_2\right)^2}.$$

8. The Lin-Tsien equation

The Lin-Tsien equation [8]

$$2\phi_{tx}+\phi_x\phi_{xx}-\phi_{yy}=0, \tag{8.1}$$

where ϕ is a velocity potential, has been used to study dynamic transonic flow in two space dimensions (x and y). The full group is known (see Ovsiannikov [3, p. 388]) to be

$$X=4\alpha x/3+g'(t)y+w(t),\quad Y=\alpha y+g(t),\quad T=2\alpha t/3+\beta, \tag{8.2}$$

$$\Phi=2\alpha\phi+2y^3g'''/3+2y^2w''+2xyg''+2xw'+yr(t)+s(t).$$

Equations (8.2) contain two arbitrary constants, α and β, and four arbitrary functions $g(t)$, $w(t)$, $r(t)$ and $s(t)$. Consequently, the Lie algebra is infinite-dimensional. Once again these arbitrary functions will permit an infinite number of time-dependent solutions to be generated from each steady-state solution.

The invariants of the group that are constant in time (with $\alpha = 0$) are found as before to be

$$x - \bar{x} = gy - \lambda, \quad \lambda = \int (g^2 + w)\, dt,$$

$$y - \bar{y} = Q, \quad Q = \int g(t)\, dt$$

and

$$\begin{aligned} \phi - \bar{\phi} = {} & 2g'\bar{x}\bar{y} + 2g''\bar{y}^3/3 + 2w\bar{x} \\ & + \int \{ 2g''[g\bar{y}^2 + 2gQ\bar{y} - \lambda\bar{y} + gQ^2 - \lambda Q] \\ & \quad + \tfrac{2}{3}g'''[3\bar{y}^2 Q + 3\bar{y}Q^2 + Q^3] + 2w'[\bar{y}g + gQ - \lambda] \\ & \quad + 2w''[2\bar{y}Q + Q^2] + r\bar{y} + s \}\, dt. \end{aligned} \tag{8.3}$$

It is easy to show that $\bar{\phi}(x, y)$ satisfies the time-independent equation

$$\bar{\phi}_{\bar{x}}\bar{\phi}_{\bar{x}\bar{x}} - \bar{\phi}_{\bar{y}\bar{y}} = 0 \tag{8.4}$$

which has been much studied. Given any solution of (8.4) it follows that time-dependent solutions (with four arbitrary functions of time g, w, r and s) are constructable, thus,

$$\begin{aligned} \phi(x, y, t) = {} & \bar{\phi}(x - g(t)y + \lambda, y - Q) \\ & + (\text{right-hand side of the last equation in (8.3)}). \end{aligned}$$

References

[1] W.F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Vol. II, Chapter 2, Academic Press, New York (1972).

[2] G.W. Bluman and J.D. Cole, *Similarity Methods for Differential Equations*, Springer, New York (1974).

[3] L.V. Ovsiannikov, *Group Analysis of Differential Equations* (Russian Edition, NAUKA 1978); English translation edited by W.F. Ames, Academic Press (1982).

[4] R.E. Boisvert, *Group Analysis of the Navier-Stokes Equations*, Ph.D. Dissertation, Georgia Institute of Technology, Atlanta, GA 30332, 1982.

[5] R.E. Boisvert, W.F. Ames and U.N. Srivastava, Group properties and new solutions of Navier-Stokes equations, *Journal of Engineering Math.* 17 (1983) 203–221.

[6] M.C. Nucci, Group analysis for MHD equations, *Atti Sem. Mat. Fis. Univ. Modena* 33 (1984) (in press).

[7] C. Rogers and W.F. Shadwick, *Bäcklund Transformations and Their Applications*, Academic Press, N.Y. (1982).

[8] K. Oswatitsch (Editor), *Symposium Transonicum (Aachen, 1962)*, Springer-Verlag Berlin, (1964).

[9] F. Schwarz. A reduce package for determining Lie symmetries of ordinary and partial differential equations, *Comp. Physics Commun.* 27 (1982) 179–186.

[10] P. Roseneau and J.L. Schwarzmeier, Similarity solutions of systems of partial differential equations using MACSYMA, Courant Inst. of Math. Sci. Report No. C00-3077-160/MF-94 (1979).